

# Topology of the Spaces of Functions with Prescribed Singularities on Surfaces

E. A. Kudryavtseva<sup>1,\*</sup>

<sup>1</sup>*Moscow State University*

Let  $M$  be a smooth connected orientable closed surface and  $f_0 \in C^\infty(M)$  a function having only critical points of the  $A_\mu$ -types,  $\mu \in \mathbb{N}$ . Let  $\mathcal{F} = \mathcal{F}(f_0)$  be the set of functions  $f \in C^\infty(M)$  having the same types of local singularities as those of  $f_0$ . We describe the homotopy type of the space  $\mathcal{F}$ , endowed with the  $C^\infty$ -topology, and its decomposition into orbits of the action of the group of “left-right changings of coordinates”.

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Let us give a short historical overview, mostly for the case of a Morse function  $f_0$  (see the paper [3] and references therein). A. T. Fomenko posed the question (1997) whether the space  $\mathcal{F}$  is arcwise connected; it was answered affirmatively by the author [6] for  $M = S^2, \mathbb{R}P^2$ , by S. V. Matveev [6] and H. Zieschang in the general case. Open  $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ -orbits in  $\mathcal{F}$  were counted by V. I. Arnold [7] and E. V. Kulinich (1998). Homotopy type of any  $\mathcal{D}^0(M)$ -orbit in  $\mathcal{F}$  was studied by S. I. Maksymenko [8] (when  $f_0$  was allowed to have certain degenerate types singularities) and by the author [3–5]. V. A. Vassiliev [9] proved the parametric  $h$ -principle and studied cohomology of spaces of smooth  $\mathbb{R}^N$ -valued functions not having too complicated singularities on any smooth manifold  $M$ . However the 1-parameter  $h$ -principle

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\* Electronic address: [eakudr@mech.math.msu.su](mailto:eakudr@mech.math.msu.su)

fails for the spaces of Morse functions on some  $M$  with  $\dim M > 5$  [10].

## 1. MAIN RESULT

For any function  $f \in C^\infty(M)$ , denote by  $C_f$  the set of its critical points, and by  $C_f^{\text{triv}}$  the set of critical points of the  $A_{2m}$ -types,  $m \in \mathbb{N}$ . Recall that, in a neighbourhood of such a point  $x \in C_f$ , there exist local coordinates  $u, v$  such that  $f = \eta(u^{2m+1} + v^2) + f(x)$  for some sign  $\eta \in \{+, -\}$ . The integer  $\eta m$  will be called the *level* of the point  $x$ .

Denote by  $C_f^{\min}$  and  $C_f^{\max}$  (respectively  $C_f^{\text{saddle}}$ ) the set of critical points of  $f$  of  $A_{2m-1}$ -types,  $m \in \mathbb{N}$ , which are (respectively are not) points of local minima or local maxima. In a neighbourhood of such a point  $x$ , there exist local coordinates  $u, v$  such that  $f = \eta(u^{2m} \pm v^2) + f(x)$  where  $\eta \in \{+, -\}$ . The integer  $\eta(m-1)$  will be called the *level* of the point  $x$ . The subset of degenerate critical points (i.e. those of non-zero levels) in  $C_f^{\text{extr}} := C_f^{\min} \cup C_f^{\max}$  will be denoted by  $\widehat{C}_f^{\text{extr}}$ .

Suppose that an action of a group  $G$  on a topological space  $X$ , a stratified [11] orbifold  $Y$  and a continuous surjection  $\varkappa : X \rightarrow Y$  are given. If every  $G$ -orbit in  $X$  is the full pre-image of a stratum from  $Y$ , we will say that  $\varkappa$  *classifies*  $G$ -orbits, while  $Y$  and  $\varkappa$  are the *classifying space* and *map*.

The group  $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$  acts on  $M \times \mathcal{F}$  by the homeomorphisms  $(x, f) \mapsto (h^{-1}(x), h_1^{-1} \circ f \circ h)$ ,  $(h_1, h) \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ . Define the *evaluation functional*  $\text{Eval} : M \times \mathcal{F} \rightarrow \mathbb{R}$ ,  $(x, f) \mapsto f(x)$ , and

$$s := \max\{0, \chi(M) + 1\} > \chi(M). \quad (1)$$

**Theorem.** *For every function  $f_0 \in C^\infty(M)$  whose all critical points are of the  $A_\mu$ -types,  $\mu \in \mathbb{N}$ , there exist smooth manifolds  $\mathcal{B}$  and  $\mathcal{E}$  and surjective submersions  $k : \mathcal{F} \rightarrow \mathcal{B}$ ,  $\varkappa : M \times \mathcal{F} \rightarrow \mathcal{E}$ ,  $\pi : \mathcal{E} \rightarrow \mathcal{B}$ ,  $\varepsilon : \mathcal{E} \rightarrow \mathbb{R}$  such that the diagram*

$$\begin{array}{ccccc} & & \text{Eval} & & \\ & & \curvearrowright & & \\ M \times \mathcal{F} & \xrightarrow{\varkappa} & \mathcal{E} & \xrightarrow{\varepsilon} & \mathbb{R} \\ \text{Pr} \downarrow & & \downarrow \pi & & \\ \mathcal{F} & \xrightarrow{k} & \mathcal{B} & & \end{array}$$

*commutes, where  $\text{Pr} : M \times \mathcal{F} \rightarrow \mathcal{F}$  is the projection and  $\dim \mathcal{B} = 2s + 2|C_{f_0}^{\text{triv}}| + |C_{f_0}^{\text{extr}}| + |\widehat{C}_{f_0}^{\text{extr}}| + 3|C_{f_0}^{\text{saddle}}| = \dim \mathcal{E} - 2$ . Moreover:*

(a) the maps  $k, \varkappa$  are homotopy equivalences and classify  $\mathcal{D}^0(M)$ - and  $\mathcal{D}(\mathbb{R}) \times \mathcal{D}^0(M)$ -orbits in  $\mathcal{F}, M \times \mathcal{F}$  for some stratifications on  $\mathcal{B}, \mathcal{E}$  whose all strata are submanifolds; the map  $\pi$  is a fibre bundle with fibres diffeomorphic to  $M$ ;

(b) the map  $k$  (resp.  $\varkappa$ ) induces a homotopy equivalence between every  $\mathcal{D}^0(M)$ -invariant subset  $B \subseteq \mathcal{F}$  (resp.  $E \subseteq M \times \mathcal{F}$ ) and its image, e.g. between every orbit from item (a) and the corresponding stratum;

(c) the group  $\mathcal{MCG}(M) = \mathcal{D}(M)/\mathcal{D}^0(M)$  discretely acts on  $\mathcal{B}, \mathcal{E}$  by diffeomorphisms preserving the stratifications from item (a) and the function  $\varepsilon$ ; the maps  $p \circ k : \mathcal{F} \rightarrow \mathcal{B}' := \mathcal{B}/\mathcal{MCG}(M)$  and  $P \circ \varkappa : M \times \mathcal{F} \rightarrow \mathcal{E}' := \mathcal{E}/\mathcal{MCG}(M)$  classify  $\mathcal{D}(M)$ - and  $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ -orbits in  $\mathcal{F}$  and  $M \times \mathcal{F}$  for the induced stratifications on  $\mathcal{B}'$  and  $\mathcal{E}'$ , where  $p : \mathcal{B} \rightarrow \mathcal{B}'$  and  $P : \mathcal{E} \rightarrow \mathcal{E}'$  are the projections.

Let us explain the term “submersion” in the case of functional spaces. If  $Q, R$  are smooth manifolds and  $\mathcal{Q} := Q \times \mathcal{F}$ , denote by  $C^\infty(R, \mathcal{Q})$  the preimage of  $C^\infty(R, Q) \times C^\infty(R \times M)$  under the inclusion  $C(R, \mathcal{Q}) \hookrightarrow C(R, Q) \times C(R \times M)$ , and by  $C^\infty(\mathcal{Q}, R)$  the set of maps inducing maps  $C^\infty(\mathbb{R}^n, \mathcal{Q}) \rightarrow C^\infty(\mathbb{R}^n, R)$  for all  $n \in \mathbb{N}$ . A map  $p \in C^\infty(\mathcal{Q}, R)$  will be called a *submersion* if, for any  $q \in \mathcal{Q}$ , there exist a neighbourhood  $U$  of the point  $p(q)$  in  $R$  and a map  $\sigma \in C^\infty(U, \mathcal{Q})$  such that  $p \circ \sigma = \text{id}_U$ .

## 2. CONSTRUCTING THE CLASSIFYING MANIFOLDS AND MAPS

Similarly to [12], by a *framed function* on an oriented surface  $M$  we will mean a pair  $(f, \alpha)$  where  $f \in C^\infty(M)$  has only the  $A_\mu$ -types local singularities and  $\alpha$  is a closed 1-form on  $M \setminus C_f^{\text{extr}}$  such that (i) the 2-form  $df \wedge \alpha$  has no zeros on  $M \setminus C_f$  and defines a positive orientation, (ii) in a neighbourhood of every critical point  $x \in C_f$  there exist local coordinates  $u, v$  such that either  $f = \eta(u^{2m+1} + v^2) + f(x)$  and  $\alpha = \eta d(v - uv)$ , or  $f = \eta(u^{2m} - v^2) + f(x)$  and  $\alpha = \eta d(uv)$ , or  $f = \eta(u^{2m} + v^2) + f(x)$  and  $\alpha = \eta \varkappa_{f,x} \frac{u dv - v du}{u^2 + v^2}$  where  $\varkappa_{f,x} = \text{const} > 0$ ,  $m \in \mathbb{N}$ ,  $\eta \in \{+, -\}$ .

Denote by  $\mathbb{F} = \mathbb{F}(f_0)$  the space of framed functions  $(f, \alpha)$  such that  $f \in \mathcal{F}$ . Endow this space with the  $C^\infty$ -topology [12]. Consider the right actions of  $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$  on  $\mathbb{F}$  and  $M \times \mathbb{F}$  by the homeomorphisms  $(f, \alpha) \mapsto (h_1^{-1} \circ f \circ h, h^* \alpha)$  and  $(x, f, \alpha) \mapsto (h^{-1}(x), h_1^{-1} \circ f \circ h, h^* \alpha)$ ,  $(h_1, h) \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ .

Let  $x_1, x_2, \dots \in M$  be pairwise distinct points. Denote by  $\mathcal{D}_r^0(M)$  the identity component

of the group  $\mathcal{D}_r(M) := \{h \in \mathcal{D}(M) \mid h(x_i) = x_i, 1 \leq i \leq r\}$ ,  $r \in \mathbb{Z}_+$ , whence  $\mathcal{D}_0(M) = \mathcal{D}(M)$ .

Define the classifying manifolds  $\mathcal{B}$  and  $\mathcal{E}$  as  $\mathcal{B} := \mathcal{B}_s$ ,  $\mathcal{E} := \mathcal{E}_s$ , where  $\mathcal{B}_r$  and  $\mathcal{E}_r$  are the *universal moduli spaces*

$$\mathcal{B}_r := \mathbb{F}/\mathcal{D}_r^0(M), \quad \mathcal{E}_r := (M \times \mathbb{F})/\mathcal{D}_r^0(M)$$

of framed functions (resp. framed functions with one marked point) in  $\mathcal{F}$ ,  $r \in \mathbb{Z}_+$ . One shows similarly to [3, 4] that  $\mathcal{B}_r$  and  $\mathcal{E}_r$  are orbifolds of dimensions  $\dim \mathcal{B}_r = 2r + 2|C_{f_0}^{\text{triv}}| + |C_{f_0}^{\text{extr}}| + |\widehat{C}_{f_0}^{\text{extr}}| + 3|C_{f_0}^{\text{saddle}}| = \dim \mathcal{E}_r - 2$ . For every group  $\mathcal{G} \in \{\mathcal{D}^0(M), \mathcal{D}(\mathbb{R}) \times \mathcal{D}^0(M)\}$ , we endow  $\mathcal{B}_r$  and  $\mathcal{E}_r$  with the stratifications whose every stratum is the full preimage of a point under the projection  $\mathcal{B}_r \rightarrow \mathcal{F}/\mathcal{G}$  and  $\mathcal{E}_r \rightarrow (M \times \mathcal{F})/\mathcal{G}$ .

Due to the  $\mathcal{D}(M)$ -equivariance of the projection  $M \times \mathbb{F} \rightarrow \mathbb{F}$  and the  $\mathcal{D}(M)$ -invariance of the evaluation functional  $M \times \mathbb{F} \rightarrow \mathbb{R}$ ,  $(x, f, \alpha) \mapsto f(x)$ , they induce some maps  $\pi_r : \mathcal{E}_r \rightarrow \mathcal{B}_r$  and  $\varepsilon_r : \mathcal{E}_r \rightarrow \mathbb{R}$ . Put  $\pi = \pi_s$ ,  $\varepsilon = \varepsilon_s$ .

Similarly to [12, Theorem 2.5] and [3, Statement 5.3], one proves the following lemmata which readily imply the theorem.

**Lemma 1.** *The projection  $\text{Forg} : \mathbb{F} \rightarrow \mathcal{F}$ ,  $(f, \alpha) \mapsto f$ , is a homotopy equivalence and has a homotopy inverse map  $i : \mathcal{F} \rightarrow \mathbb{F}$  and corresponding homotopies that respect the projections  $q : \mathcal{F} \rightarrow \mathcal{F}/\mathcal{D}^0(M)$  and  $q \circ \text{Forg} : \mathbb{F} \rightarrow \mathcal{F}/\mathcal{D}^0(M)$ .*

**Lemma 2.** *If  $r \geq s$  then  $\mathcal{B}_r$  is a smooth manifold, while the projection  $\text{Ev}_r : \mathbb{F} \rightarrow \mathcal{B}_r$  is a homotopy equivalence and has a homotopy inverse map  $i_r : \mathcal{B}_r \rightarrow \mathbb{F}$  and corresponding homotopies that respect  $\text{Ev}_r$  (whence  $\text{Ev}_r \circ i_r = \text{id}_{\mathcal{B}_r}$ ).*

Put  $k_r = \text{Ev}_r \circ i : \mathcal{F} \rightarrow \mathcal{B}_r$ . One defines similarly  $\varkappa_r$ . Define the classifying maps  $k = k_s$ ,  $\varkappa = \varkappa_s$ .

### 3. REDUCING TO THE CASE OF MORSE FUNCTIONS

If  $f_0$  is a Morse function and  $s = 0$ , then the space  $\mathcal{B}$  from §2 coincides with the smooth stratified manifold  $\widetilde{\mathcal{M}}$  (the *universal moduli space of framed Morse functions*) studied in [3–5]. It happens that every  $\mathcal{B}_r$  and  $\mathcal{E}_r$  can be described in terms of Morse functions.

Recall that a function  $f \in C^\infty(M)$  is said to be *Morse* if all its critical points are nondegenerate (i.e. have the  $A_1$ -type, cf. §1). Denote by  $\text{Morse}(f_0)$  the space of Morse functions on

$M$  having exactly  $|C_{f_0}^{\min}|$  and  $|C_{f_0}^{\max}|$  points of local minima and maxima and  $|C_{f_0}^{\text{saddle}}|$  saddle points.

A Morse function  $f \in \text{Morse}(f_0)$  will be called  $f_0$ -labeled if every its critical point  $x \in C_f$  is labeled by an integer and, in the case when this integer does not vanish and  $x \in C_f^{\text{extr}}$ , also by a 1-dimensional subspace  $\ell_x \subset T_x M$ , moreover  $|C_{f_0}^{\text{triv}}|$  of non-critical points of  $f$  are labeled by non-zero integers in such a way that the level (cf. §1) of every critical point of  $f_0$  coincides with the integer label of the corresponding labeled point of  $f$ , for some bijections  $C_{f_0}^{\min} \approx C_f^{\min}$ ,  $C_{f_0}^{\max} \approx C_f^{\max}$ ,  $C_{f_0}^{\text{saddle}} \approx C_f^{\text{saddle}}$  and a bijection between  $C_{f_0}^{\text{triv}}$  and the set of labeled non-critical points of  $f$ .

Denote by  $\text{Morse}^*(f_0)$  the space of framed (cf. §2)  $f_0$ -labeled Morse functions. It is not difficult to construct homeomorphisms

$$\mathcal{B}_r \approx \text{Morse}^*(f_0)/\mathcal{D}_r^0(M), \quad \mathcal{E}_r \approx (M \times \text{Morse}^*(f_0))/\mathcal{D}_r^0(M), \quad r \in \mathbb{Z}_+. \quad (2)$$

#### 4. RELATION WITH MEROMORPHIC FUNCTIONS AND THE CONFIGURATION SPACES

Suppose that  $M$  is either a sphere  $S^2$  or a torus  $T^2$ . If  $M = S^2$ , denote by  $\mathbb{A}(f_0)$  the space of rational functions  $R$  on the Riemann sphere  $\overline{\mathbb{C}}$  such that all poles of the 1-form  $\omega = R(z)dz$  are simple and have real residues, being positive at  $|C_{f_0}^{\min}|$  poles and negative at  $|C_{f_0}^{\max}|$  poles. If  $M = T^2$ , denote by  $\mathbb{A}(f_0)$  the space of pairs  $(\lambda, R)$  where  $\lambda \in \mathbb{C}$ ,  $\text{Im } \lambda > 0$ , and  $R$  is a meromorphic function on the torus  $T_\lambda^2 = \mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z})$ , whose poles are all simple, all periods of the meromorphic 1-form  $\omega = R(z)dz$  are purely imaginary, and the residues are positive at  $|C_{f_0}^{\min}|$  poles and negative at  $|C_{f_0}^{\max}|$  poles.

Let  $\mathbb{A}_0(f_0)$  be the space of functions  $R \in \mathbb{A}(f_0)$  or pairs  $(\lambda, R) \in \mathbb{A}(f_0)$  such that  $\omega = R(z)dz$  has only simple zeros.

Due to [13, Proposition 3.4], the assignment to a 1-form  $\omega$  its poles and residues at them gives a bijection  $\varphi : \mathbb{A}(f_0) \xrightarrow{\sim} C(f_0)$ , where  $C(f_0)$  is the “labeled configuration space” consisting of  $|C_{f_0}^{\text{extr}}|$ -points subsets of  $M$  equipped by  $|C_{f_0}^{\min}|$  positive and  $|C_{f_0}^{\max}|$  negative real marks with zero total sum. Thus  $\mathbb{A}_0(f_0)$  is homeomorphic to the open subset  $\varphi(\mathbb{A}_0(f_0)) \subseteq C(f_0)$  consisting of the “labeled configurations” that correspond to 1-forms  $\omega$  without multiple zeros.

It is not difficult to derive from (2) with  $r = s$  (cf. (1) and [12, Remark 2.6]) that our manifold  $\mathcal{B}$  is homeomorphic to the space  $\mathbb{A}_0^*(f_0)$  of functions  $R \in \mathbb{A}_0(f_0)$  or pairs  $(\lambda, R) \in \mathbb{A}_0(f_0)$ , marked by  $f_0$ -labels (cf. §3) at zeros and poles of the 1-form  $\omega = R(z)dz$  and at some other  $|C_{f_0}^{\text{triv}}|$  points, as well as by a “vertical” label consisting of (i) a real label and (ii) either a positive real label in the case of  $|C_{f_0}^{\text{triv}}| = |C_{f_0}^{\text{saddle}}| = 0$ , or  $|C_{f_0}^{\text{extr}}|$  integral curves of the field  $\ker(\text{Re } \omega)$  separating the poles from other labeled points. Thus, the manifold  $\mathcal{B} \approx \mathbb{A}_0^*(f_0)$  can be obtained from the “labeled configuration subspace”  $\varphi(\mathbb{A}_0(f_0)) \subseteq C(f_0)$  by assigning the  $f_0$ -labels and the (topologically inessential) “vertical” label.

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1. *Kudryavtseva E. A.* The topology of spaces of functions with prescribed singularities on surfaces // Proc. Int. Conf. “XVII Geometrical Seminar” (Zlatibor, Sept. 3–8, 2012). Beograd: Matematički fakultet, 2012. 45–47. <http://poincare.matf.bg.ac.rs/geometricalseminar/abstracts>
  2. *Kudryavtseva E. A.* Topology and stratification of spaces of functions with prescribed singularities on surfaces // Proc. Int. Conf. “Analysis and singularities” (Moscow, Dec. 17–21, 2012). Moscow: Steklov Math. Inst. RAS, 2012. 141–143. <http://arnold75.mi.ras.ru/Abstracts.pdf>
  3. *Kudryavtseva E. A.* The topology of spaces of Morse functions on surfaces // Math. Notes **92**:2 (2012), 219–236. arXiv:1104.4792
  4. *Kudryavtseva E. A.* On the homotopy type of the spaces of Morse functions on surfaces // Sborn. Math. **204**:1 (2013), 75–113. arXiv:1104.4796
  5. *Kudryavtseva E. A.* Special framed Morse functions on surfaces // Moscow Univ. Math. Bull. **67**:4 (2012), 151–157. arXiv:1106.3116.
  6. *Kudryavtseva E. A.* Realization of smooth functions on surfaces as height functions // Sb. Math. **190**:3 (1999), 349–405.

7. *Arnold V. I.* Topological classification of Morse functions and generalisations of Hilbert's 16-th problem // Math. Phys. Anal. Geom. **10**:3 (2007), 227–236.
8. *Maksymenko S. I.* Homotopy types of stabilizers and orbits of Morse functions on surfaces // Ann. Glob. Anal. Geom. **29**:3 (2006), 241–285. arXiv:0310067
9. *Vasil'ev V. A.* Topology of spaces of functions without compound singularities // Funct. Anal. Appl. **23**:4 (1989), 277–286.
10. *Chenciner A., Laudenbach F.* Morse 2-jet space and  $h$ -principle // Bull. Brazil. Math. Soc. **40**:4 (2009), 455–463. arXiv:0902.3692
11. *Whitney H.* Tangents to an analytic variety // Ann. Math. **81**:3 (1965), 496–549.
12. *Kudryavtseva E. A., Permyakov D. A.* Framed Morse functions on surfaces // Sborn. Math. **201**:4 (2010), 501–567.
13. *Grushevsky S., Krichever I.* The universal Whitham hierarchy and geometry of the moduli space of pointed Riemann surfaces // In: Surveys in Differ. Geom. **14** (2010). Int. Press, Somerville, MA. 111–129. arXiv:0810.2139